ACES School of Electromagnetics
Online Lectures
February 21, 2026

Microwave and Millimeter-wave Imaging in Real Time

Part 2: Models of Electromagnetic Scattering

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LECTURE OVERVIEW

Part 1: Introduction
- Applications of Microwave and mm-Wave Imaging (MMI)
- Components of MMI Systems and Their Requirements
- Data Acquisition Systems, Antennas and Antenna Arrays in MMI

Part 2: Models of Electromagnetic Scattering
- Data and State Equations
- Linearized Scattering Models

Part 3: Fourier-domain Direct Image-reconstruction Methods
- Core Concept: Data Point-spread Function (PSF)
- Quantitative Microwave Holography (QMH)
- Scattered Power Mapping (SPM)

FIELD-BASED FORWARD MODELS: *DATA* and *STATE* EQUATIONS

- **data equation:** maps contrast to data (field measured outside OUT)

$$\underbrace{\mathbf{E}^{\mathrm{sc}}(\mathbf{r}\in S_{\mathrm{a}})}_{\text{data } \boldsymbol{d}} \approx \left[\mathbf{E}^{\mathrm{tot}}-\mathbf{E}^{\mathrm{inc}}\right]_{\mathbf{r}\in S_{\mathrm{a}}} = \iiint_{V_{\mathrm{s}}} \underset{\uparrow}{K}(\mathbf{r}')\underline{\underline{\mathbf{G}}}_{\mathrm{b}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}^{\mathrm{tot}}(K,\mathbf{r}')d\mathbf{r}' \qquad \boxed{\mathbf{r}\notin V_{\mathrm{s}}}$$

contrast $K(\mathbf{r}') = k_{\mathrm{s}}^{2}(\mathbf{r}') - k_{\mathrm{b}}^{2}$

  - ➢ ensures that the contrast produces scattered field matching the measurements

- **state equation:** maps the contrast to the field inside OUT

$$\mathbf{E}^{\mathrm{tot}}(\mathbf{r}\in V_{\mathrm{s}}) = \iiint_{V_{\mathrm{s}}} K(\mathbf{r}')\underline{\underline{\mathbf{G}}}_{\mathrm{b}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}^{\mathrm{tot}}(K,\mathbf{r}')d\mathbf{r}' \qquad \boxed{\mathbf{r}\in V_{\mathrm{s}}}$$

  - ➢ ensures that the internal field satisfies Maxwell's equations for the contrast found from the data equation

- image reconstruction is an interplay of the two forward-model equations
- the complete inverse problem solution involves determining not only the contrast but also the total internal field

FIELD-BASED DATA EQUATION: A CLOSER LOOK

$$\mathbf{E}^{\mathrm{sc}}(\mathbf{r}\in S_{\mathrm{a}}) = \left[\mathbf{E}^{\mathrm{tot}} - \mathbf{E}^{\mathrm{inc}}\right]_{\mathbf{r}\in S_{\mathrm{a}}} = \iiint_{V_{\mathrm{s}}} K(\mathbf{r}')\underline{\underline{\mathbf{G}}}_{\mathrm{b}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}^{\mathrm{tot}}(K,\mathbf{r}')d\mathbf{r}'$$

$\Downarrow$

data

Q1: Do we measure the external (data) **E**-fields (total and incident)?

No, we measure S-parameters or voltage waveforms

Q2: Do we know Green’s dyadic $\underline{\underline{\mathbf{G}}}_{\mathrm{b}}(\mathbf{r},\mathbf{r}')$?

No, unless the medium is uniform (or layered) and unbounded

Q3: Do we know the total internal field $\mathbf{E}^{\mathrm{tot}}(K,\mathbf{r}')$?

No, but we can employ Born’s approximation → $\mathbf{E}^{\mathrm{tot}}(K,\mathbf{r}') \approx \mathbf{E}^{\mathrm{inc}}(\mathbf{r}')$

Q4: Do we know the incident internal field $\mathbf{E}^{\mathrm{inc}}(\mathbf{r}')$?

No, measuring it is impractical, but we can simulate it

We need a data equation that matches the actual measured responses!

DATA EQUATION IN TERMS OF SCATTERING PARAMETERS

[**Beaverstone** ***et al.***, ***IEEE Trans. MTT***, **2017**]

- scattering from penetrable objects (isotropic scattering assumed)

$$S_{ik}^{\mathrm{sc}} = \underbrace{\frac{\mathrm{i}\omega\varepsilon_0}{2a_i a_k}}_{\text{known constants}} \iiint_{V_\mathrm{s}} \Delta\varepsilon_\mathrm{r}(\mathbf{r}')\mathbf{E}_i^{\mathrm{inc}}(\mathbf{r}')\cdot\mathbf{E}_k^{\mathrm{tot}}(\mathbf{r}')d\mathbf{r}'$$

$$\Delta\varepsilon_\mathrm{r}(\mathbf{r}') = \varepsilon_\mathrm{r}(\mathbf{r}') - \varepsilon_{\mathrm{r,b}}(\mathbf{r}')$$

data

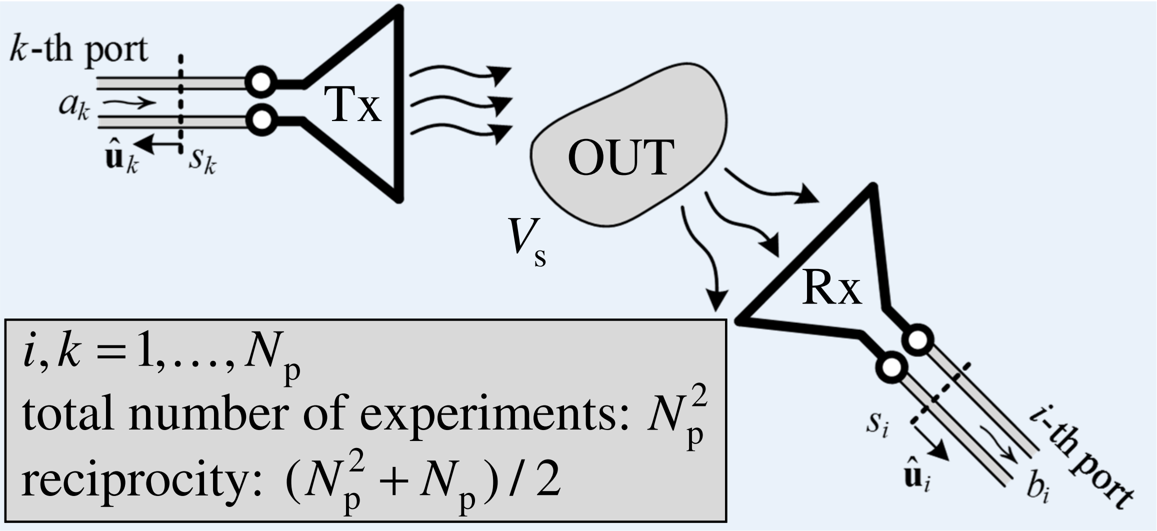
complex permittivity contrast

Green's vector function

total internal field

$\mathbf{E}_i^{\mathrm{inc}}(\mathbf{r}')$: incident internal field due to Rx antenna if it were to transmit

$\mathbf{E}_k^{\mathrm{tot}}(\mathbf{r}')$: total internal field due to Tx antenna



$i, k = 1, \ldots, N_\mathrm{p}$
total number of experiments: N_p^2
reciprocity: $(N_\mathrm{p}^2 + N_\mathrm{p})/2$

DATA EQUATION: BORN'S APPROXIMATION OF TOTAL *INTERNAL* FIELD

$$\boxed{S_{ik}^{\mathrm{sc}} = \frac{\mathrm{i}\omega\varepsilon_0}{2a_i a_k} \iiint_{V_\mathrm{s}} \Delta\varepsilon_\mathrm{r}(\mathbf{r}')\mathbf{E}_i^{\mathrm{inc}}(\mathbf{r}') \cdot \mathbf{E}_k^{\mathrm{tot}}\left(\mathbf{r}'; \Delta\varepsilon_\mathrm{r}(\mathbf{r}')\right) d\mathbf{r}'}$$

total internal field?

- the total field $\mathbf{E}_k^{\mathrm{tot}}(\mathbf{r}')$ is generally unknown AND it depends on the contrast: ***data equation is inherently nonlinear in the unknown contrast***
- 0th-order Born's approximation linearizes the data equation by replacing the unknown ***total internal field*** with the known ***incident internal field*** (Max Born, 1926)

linearized data equation

$$\boxed{\mathbf{E}_k^{\mathrm{tot}}(\mathbf{r}') \approx \mathbf{E}_k^{\mathrm{inc}}(\mathbf{r}')} \Rightarrow \boxed{S_{ik}^{\mathrm{sc}} \approx \frac{\mathrm{i}\omega\varepsilon_0}{2a_i a_k} \iiint_{V_\mathrm{s}} \Delta\varepsilon_\mathrm{r}(\mathbf{r}')\underbrace{\mathbf{E}_i^{\mathrm{inc}}(\mathbf{r}') \cdot \mathbf{E}_k^{\mathrm{inc}}(\mathbf{r}')}_{} d\mathbf{r}'}$$

data
(known)

contrast
(unknown)

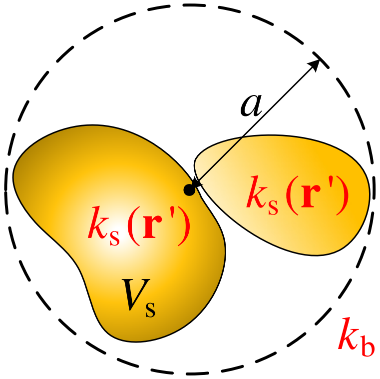
incident fields
(assumed known)

BORN'S 0th-ORDER APPROXIMATION OF THE TOTAL *INTERNAL* FIELD

- Born's 0th-order approximation of the total internal field linearizes the data equation and it is underlying all ***direct*** inversion methods (real-time imaging)
- BUT there are limits on both the size and contrast of the scatterer

$$\boxed{a^2 \left| k_\mathrm{s}^2(\mathbf{r}) - k_\mathrm{b}^2 \right| \ll 1,\ \mathbf{r} \in V_\mathrm{s}}$$

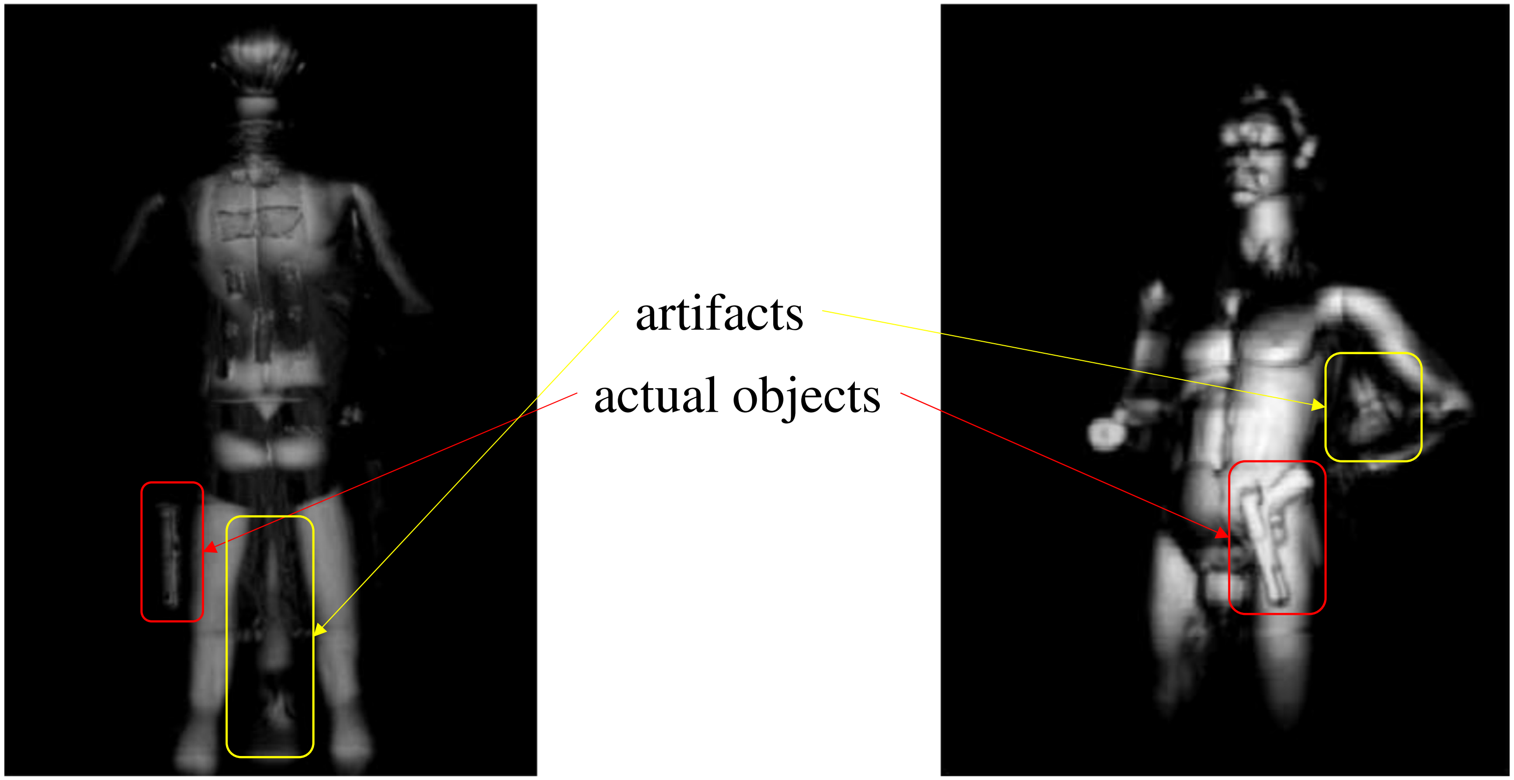
[Nikolova, *Introduction to Microwave Imaging*, 2017]



- if OUT violates the limits: images contain *artifacts* which reflect differences between $\mathbf{E}_\mathrm{Tx}^\mathrm{inc}(\mathbf{r}')$ and $\mathbf{E}_\mathrm{Tx}^\mathrm{tot}(\mathbf{r}')$ rather than contrast

IMAGE ARTIFACTS DUE TO BORN’S APPROXIMATION IN LINEARIZED MODEL

[Sheen *et al.*, *Applied Optics* 2010]



artifacts

actual objects

40 GHz to 60 GHz cylindrical scan

10 GHz to 20 GHz polarimetric cylindrical scan

BORN’S 1st-ORDER APPROXIMATION OF THE DATA (*S*-PARAMETERS)

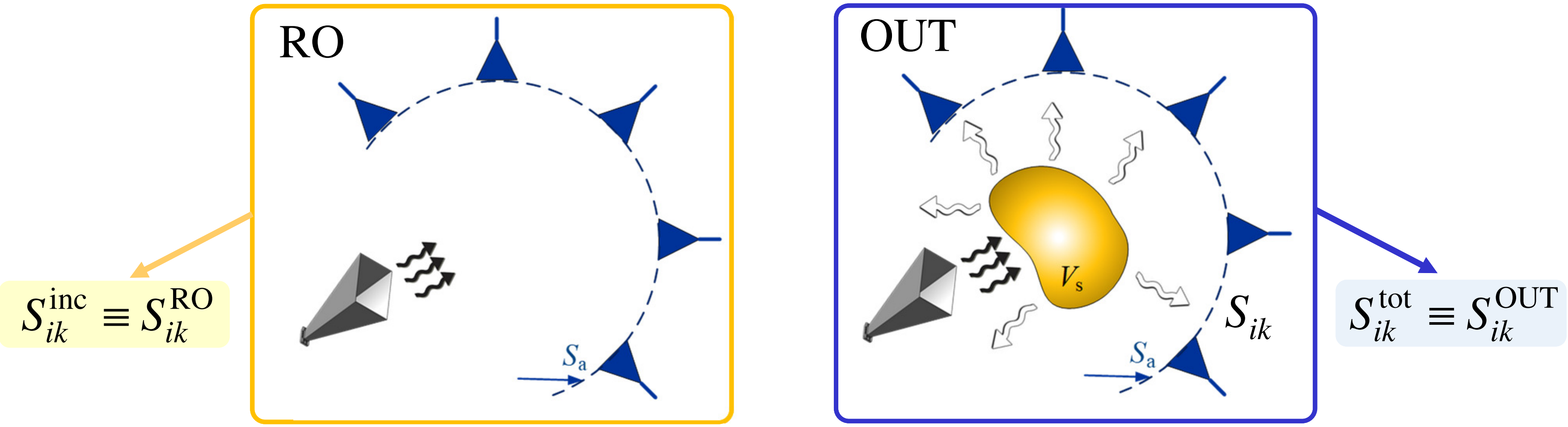
- Born’s approximation (1st order) of the ***total-field*** response: superposition of incident-field and scattered-field contributions (neglects coupling and multiple scattering between the OUT and the antennas)

$$\underset{\uparrow}{S_{ik}^{\text{tot}}} \approx \underset{\uparrow}{S_{ik}^{\text{inc}}} + \left(S_{ik}^{\text{sc}}\right)_{\text{B}} = S_{ik}^{\text{inc}} + \frac{\mathrm{i}\omega\varepsilon_0}{2a_i a_k}\iiint_{V_s} \Delta\varepsilon_{\text{r}}(\mathbf{r}')\mathbf{E}_i^{\text{inc}}(\mathbf{r}')\cdot\mathbf{E}_k^{\text{inc}}(\mathbf{r}')d\mathbf{r}'$$

total-field data *incident-field data*

Born’s approximation of the scattered-field data

- acquisition of the incident-field (background) data with *reference object* (RO) – RO is simply the setup in the absence of an OUT



$S_{ik}^{\text{inc}} \equiv S_{ik}^{\text{RO}}$

$S_{ik}^{\text{tot}} \equiv S_{ik}^{\text{OUT}}$

LIMITATIONS OF BORN'S 1st ORDER APPROXIMATION OF THE DATA

- data approximation with Born's model

$$S_{ik}^{\mathrm{sc}} \approx \left(S_{ik}^{\mathrm{sc}}\right)_{\mathrm{B}} = S_{ik}^{\mathrm{tot}} - S_{ik}^{\mathrm{inc}} \approx \frac{\mathrm{i}\omega\varepsilon_0}{2a_i a_k} \iiint_{V_{\mathrm{s}}} \Delta\varepsilon_{\mathrm{r}}(\mathbf{r}')\mathbf{E}_i^{\mathrm{inc}}(\mathbf{r}')\cdot\mathbf{E}_k^{\mathrm{inc}}(\mathbf{r}')d\mathbf{r}'$$

- limit on the BA 1st-order data approximations as related to the scattering object – less strict compared to that for the internal field

$$2a\left|k_{\mathrm{s}}(\mathbf{r}) - k_{\mathrm{b}}\right|_{\max} < \pi$$

[Slaney et al., *IEEE Trans. MTT*, 1984]

compare with $\boxed{a^2\left|k_{\mathrm{s}}^2(\mathbf{r}) - k_{\mathrm{b}}^2\right| \ll 1}$

total internal field limitation of 0th-order Born approximation

BORN'S APPROXIMATIONS EMPLOYED IN DATA EQUATION – SUMMARY

1) Born's 1st-order approximation offers simple method to extract the scattered portion of responses (data)

$$S_{ik}^{\text{sc}} = \underbrace{S_{ik}^{\text{tot}}}_{\text{OUT}} - \underbrace{S_{ik}^{\text{inc}}}_{\text{RO}}$$

NOTE: RO (reference object) is not just uniform medium – it includes all complexities of the measurement setup

2) Born's 0th-order approximation of the total internal field supplies the ***linearized model of scattering***

$$\left(S_{ik}^{\text{sc}}\right)_{\text{B}} \approx S_{ik}^{\text{sc}} = S_{ik}^{\text{tot}} - S_{ik}^{\text{inc}} \approx \frac{\text{i}\omega\varepsilon_0}{2a_i a_k} \iiint_{V_{\text{s}}} \Delta\varepsilon_{\text{r}}(\mathbf{r}')\mathbf{E}_i^{\text{inc}}(\mathbf{r}') \cdot \mathbf{E}_k^{\text{inc}}(\mathbf{r}') d\mathbf{r}'$$

NOTE: in reality

$$S_{ik}^{\text{sc}} = \frac{\text{i}\omega\varepsilon_0}{2a_i a_k} \iiint_{V_{\text{s}}} \Delta\varepsilon_{\text{r}}(\mathbf{r}')\mathbf{E}_i^{\text{inc}}(\mathbf{r}') \cdot \mathbf{E}_k^{\text{tot}}\left(\mathbf{r}'; \Delta\varepsilon_{\text{r}}(\mathbf{r}')\right) d\mathbf{r}'$$

SUMMARY OF PART 2

- forward models are essential components of imaging
- EM inverse scattering uses 2 forward-model equations
  - data equation: relates measured data to object permittivity contrast
  - state equation: relates total internal field to permittivity contrast
- we use 0th-order Born approximation (same as Rytov) to linearize the data equation

$$\boxed{\mathbf{E}_k^{\text{tot}}(\mathbf{r}') \approx \mathbf{E}_k^{\text{inc}}(\mathbf{r}')}$$

- we use 1st-order Born or Rytov approximations (or both) to extract the scattered-field data from the total-field (measured data)

1st-order BA

$$(S_{ik}^{\text{sc}})_{\text{B}} = \underbrace{S_{ik}^{\text{tot}}}_{\text{OUT}} - \underbrace{S_{ik}^{\text{inc}}}_{\text{RO}}$$

1st-order RA

[Tajik *et al.*, *Progress In Electromagn. Res. B*, 2017]
[Tajik *et al.*, *IEEE Trans. MTT* 2022]

$$(S_{ik}^{\text{sc}})_{\text{R}} \approx S_{ik}^{\text{inc}} \ln\left(\frac{S_{ik}^{\text{tot}}}{S_{ik}^{\text{inc}}}\right)$$

(not discussed here)

SUMMARY OF PART 2, cont.

- ***direct inversion methods*** solve the linearized data equation only (they do not enforce the state equation) leading to superior speed (real-time imaging)
- for best accuracy, the data equation must be expressed in terms of the measured data (e.g., *S*-parameters) instead of **E**-field

$$S_{ik}^{\mathrm{sc}} \approx \frac{\mathrm{i}\omega\varepsilon_0}{2a_i a_k} \iiint_{V_{\mathrm{s}}} \Delta\varepsilon_{\mathrm{r}}(\mathbf{r}') \mathbf{E}_i^{\mathrm{inc}}(\mathbf{r}') \cdot \mathbf{E}_k^{\mathrm{inc}}(\mathbf{r}') d\mathbf{r}'$$

QUESTIONS?

REFERENCES

Beaverstone *et al*., “Integral equations of scattering for scalar frequency-domain responses,” *IEEE Trans. Microwave Theory Tech.*, vol. 64, 2017.

Nikolova, *Introduction to Microwave Imaging*. Cambridge University Press, 2017.

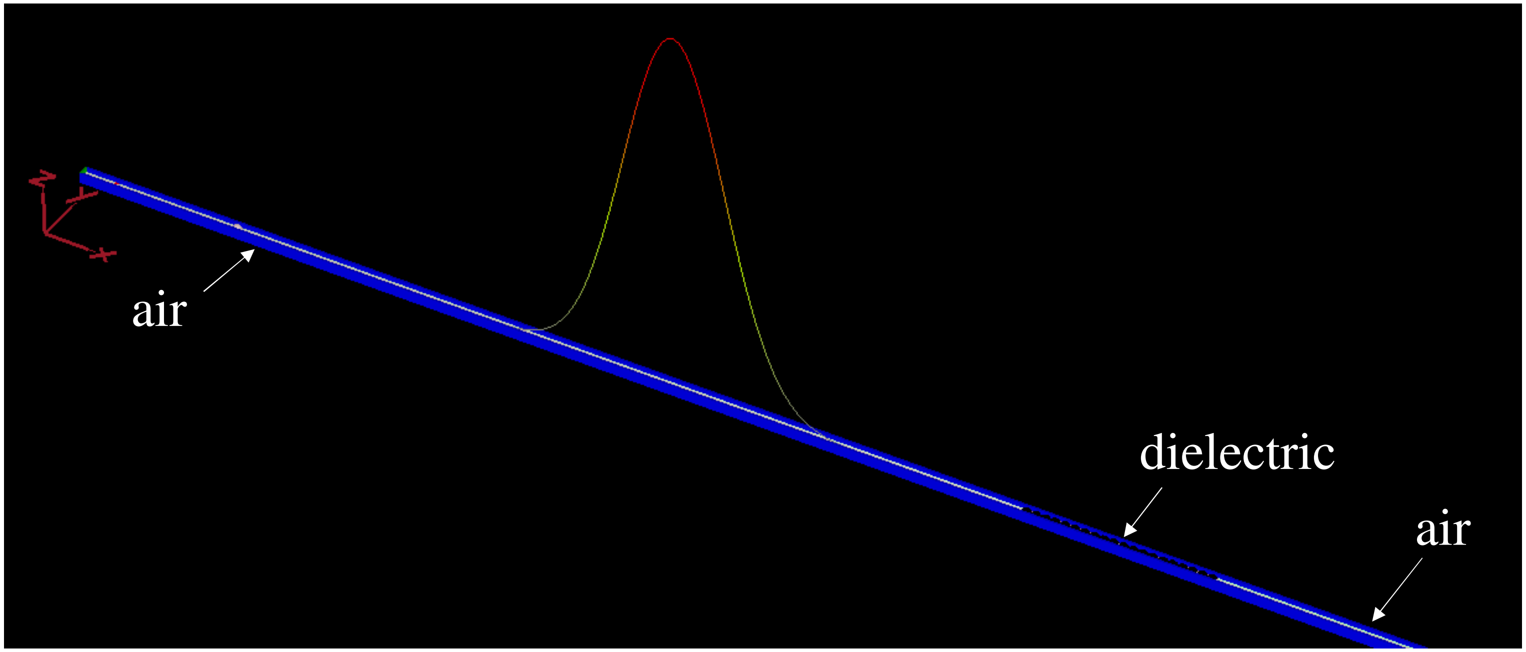
Sheen, *et al*., "Near-field three-dimensional radar imaging techniques and applications." *Applied Optics,* vol. 49, 2010.

Slaney *et al*., "Limitations of imaging with first-order diffraction tomography," *IEEE Trans. Microwave Theory Tech.*, vol. 32, 1984.

Tajik *et al*., “Real-time imaging with simultaneous use of Born and Rytov approximations in quantitative microwave holography,” *IEEE Trans. Microwave Theory Tech.*, vol. 70, 2022.

Tajik *et al*., “Comparative study of the Rytov and Born approximations in quantitative microwave holography,” *Progress in Electromagn. Res. B*, vol. 79, 2017.

1-D EXAMPLE: BORN’S APPROXIMATION OF TOTAL *INTERNAL* FIELD

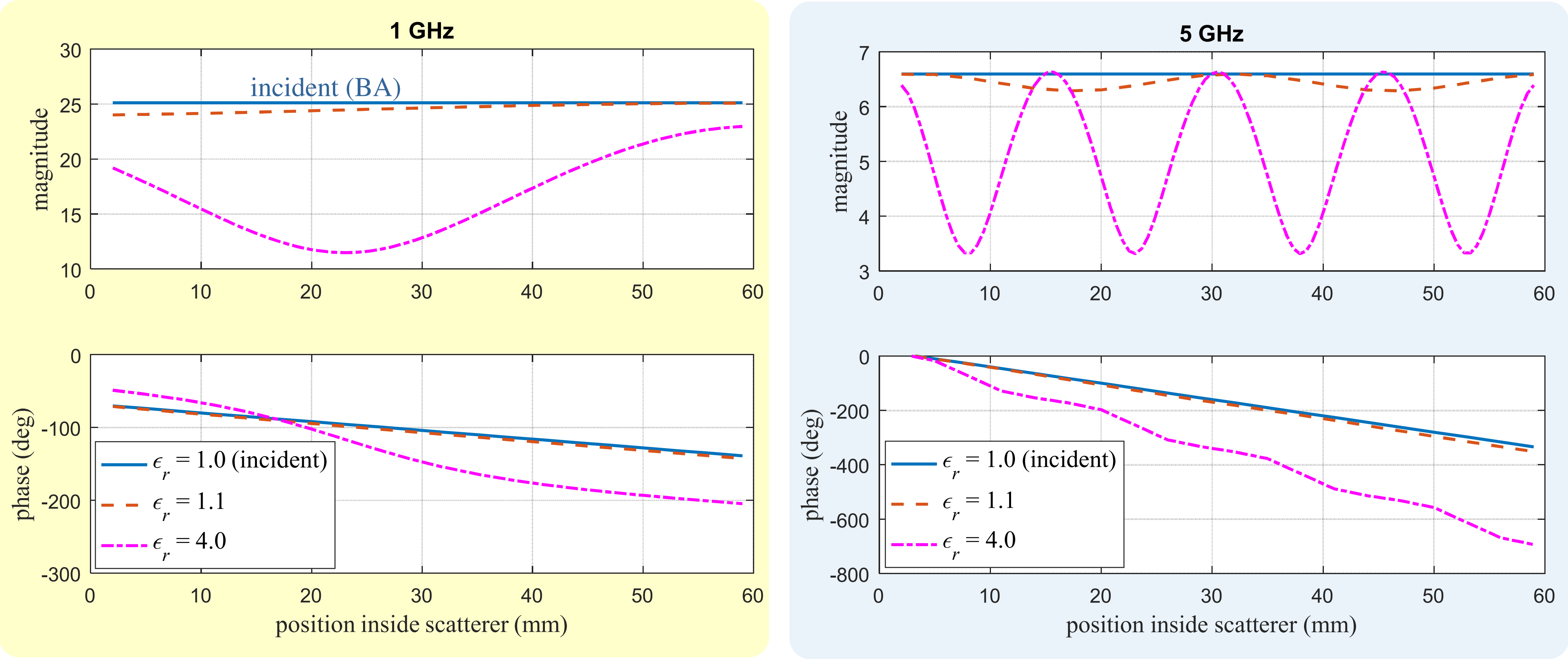


MEFiSTo-3D Nova

- Gaussian pulse bandwidth: 5 GHz at 3-dB level
- 1-D incident wave coming from left
- internal field recorded inside dielectric slab of length L = 6 cm

1-D EXAMPLE: BORN’S APPROXIMATION OF THE TOTAL ***INTERNAL*** FIELD – 2

comparison of incident wave in air (ε_r = 1.0) with actual internal field in dielectric slabs



1-D EXAMPLE: BORN'S APPROXIMATION OF THE TOTAL *INTERNAL* FIELD – 3

$a = L/2 = 3$ cm
$\varepsilon_{\mathrm{r,b}} = 1$
$\varepsilon_{\mathrm{r,s}} = 1.1,\ 4.0$
$\lambda_{\mathrm{b}}^{\mathrm{1GHz}} \approx 30$ cm
$\lambda_{\mathrm{b}}^{\mathrm{5GHz}} \approx 6$ cm

- let us evaluate Born's limit in this example – dielectric slab

$$\boxed{a^2 \left| k_{\mathrm{s}}^2(\mathbf{r}) - k_{\mathrm{b}}^2 \right| \ll 1} \;\Rightarrow\; a^2 k_{\mathrm{b}}^2 \left(\frac{k_{\mathrm{s}}^2}{k_{\mathrm{b}}^2} - 1 \right) = \left(\frac{2\pi a}{\lambda_{\mathrm{b}}} \right)^2 \left(\frac{\varepsilon_{\mathrm{r,s}}}{\varepsilon_{\mathrm{r,b}}} - 1 \right) \ll 1$$

$$\Rightarrow \begin{array}{l} \left(\varepsilon_{\mathrm{r,s}}^{\mathrm{1GHz}} \right)_{\max} < 3.53 \\ \left(\varepsilon_{\mathrm{r,s}}^{\mathrm{5GHz}} \right)_{\max} < 1.10 \end{array}$$

- BA holds marginally for slab of $\varepsilon_{\mathrm{r,s}} = 1.1$ but error is very large at $\varepsilon_{\mathrm{r,s}} = 4.0$
- BA in the **magnitude** field distribution is more sensitive (than the **phase**) to permittivity contrast because reflections at interfaces are not taken into account – even for $\varepsilon_{\mathrm{r,s}} = 1.1$ magnitude errors are appreciable (esp. at 5 GHz)
- error of the internal-field BA grows with frequency due to increase in scatterer's electrical size *a/λ*

1-D EXAMPLE: LIMITATIONS OF BORN'S APPROXIMATION OF THE DATA

- re-visiting the dielectric-slab example for scattered field at external observation points (ports 1 and 2)

- What is the contrast limit now?

$$\boxed{2a\left|k_s(\mathbf{r})-k_b\right|_{\max}<\pi} \Rightarrow \underbrace{2a}_{L}k_b\left(\frac{k_s}{k_b}-1\right)=\left(\frac{2\pi L}{\lambda_b}\right)^2\left(\sqrt{\frac{\varepsilon_{r,s}}{\varepsilon_{r,b}}}-1\right)<\pi$$

[Slaney et al., *IEEE Trans. MTT*, 1984]

$a = L/2 = 3$ cm

$\varepsilon_{r,b} = 1$

$\lambda_b^{1\text{GHz}} \approx 30$ cm

$\lambda_b^{5\text{GHz}} \approx 6$ cm

$$\Rightarrow \begin{array}{l}\left(\varepsilon_{r,s}^{1\text{GHz}}\right)_{\max} < 12.25\\ \left(\varepsilon_{r,s}^{5\text{GHz}}\right)_{\max} < 2.25\end{array}$$

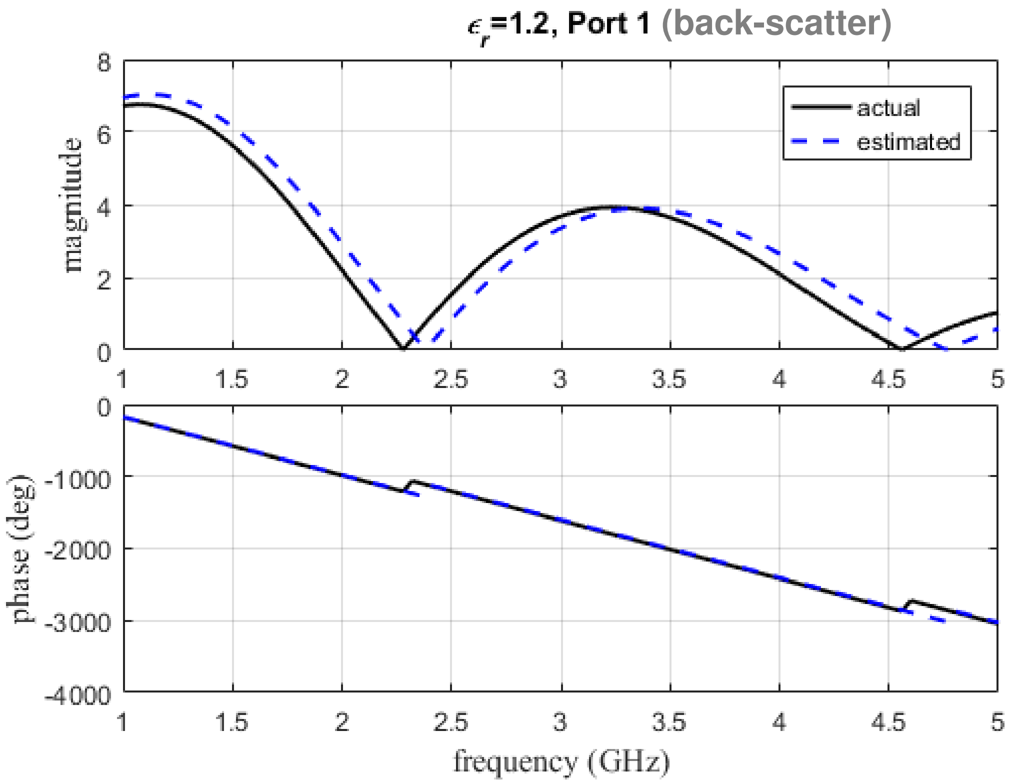
compare with internal-field BA limits

- Be aware! Slaney's limit is derived with transmission measurements in mind!

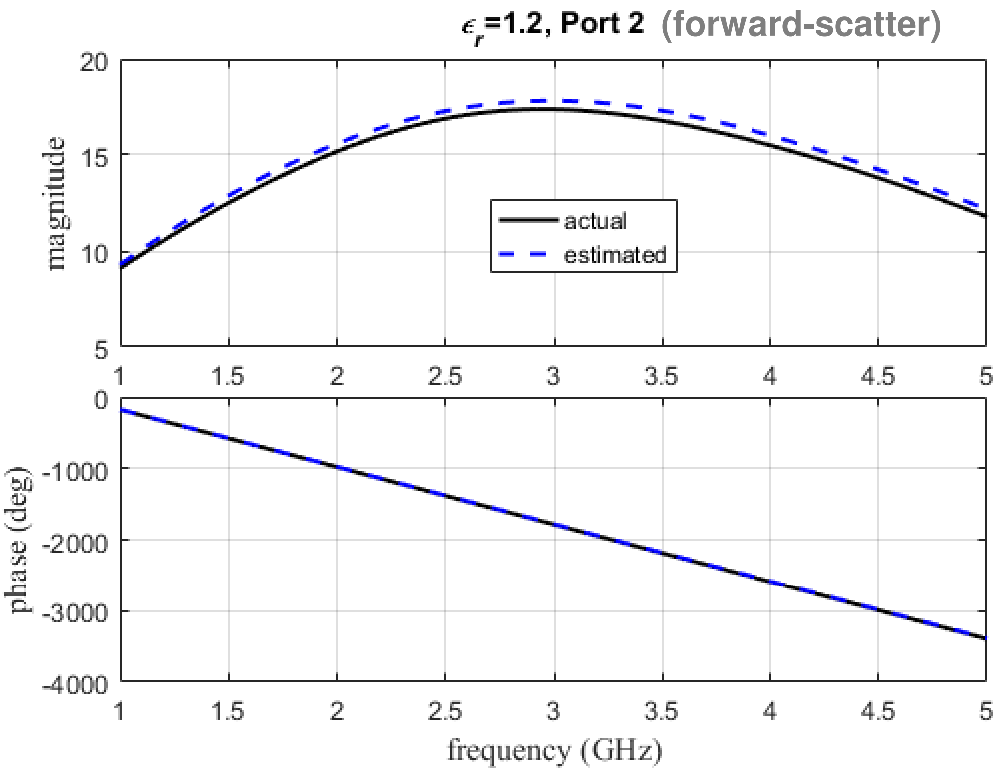
[Nikolova, *Introduction to Microwave Imaging*, 2017]

1-D EXAMPLE: LIMITATIONS OF BORN'S APPROXIMATION OF THE DATA

slab relative permittivity = 1.2



ϵ_r=1.2, Port 1 (back-scatter)

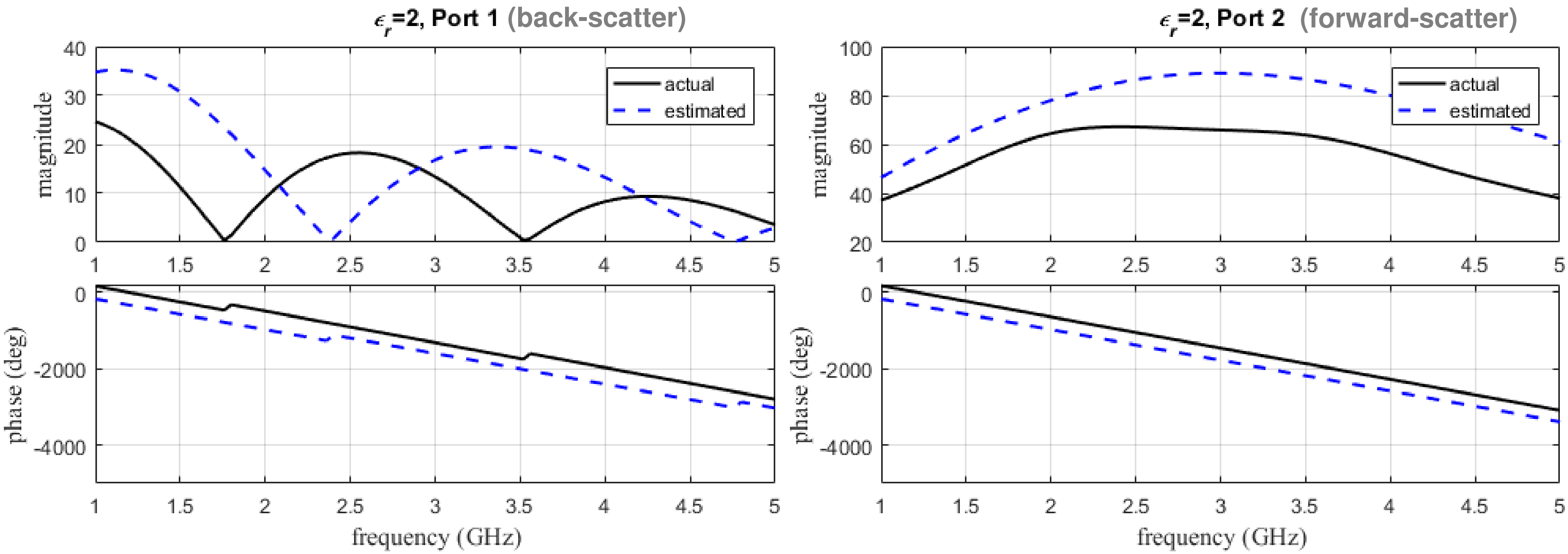


ϵ_r=1.2, Port 2 (forward-scatter)

60 MM THICK DIELECTRIC SLAB IN AIR

1-D EXAMPLE: LIMITATIONS OF BORN'S APPROXIMATION OF THE DATA

slab relative permittivity = 2



- errors at Port 1 (reflection measurement) are unacceptable, esp. magnitude

EXAMPLE: BORN vs. RYTOV APPROXIMATION OF THE DATA

- re-visiting the dielectric-slab example for scattered field at external observation points (ports 1 and 2)
- What is Rytov's contrast limit?

$$\boxed{\left(\varepsilon_{\mathrm{r,s}} - \varepsilon_{\mathrm{r,b}}\right) / \varepsilon_{\mathrm{r,b}} < 1} \quad ⇨ \quad \varepsilon_{\mathrm{r,s}} < 2\varepsilon_{\mathrm{r,b}} = 2$$

➢ compare with BA limits

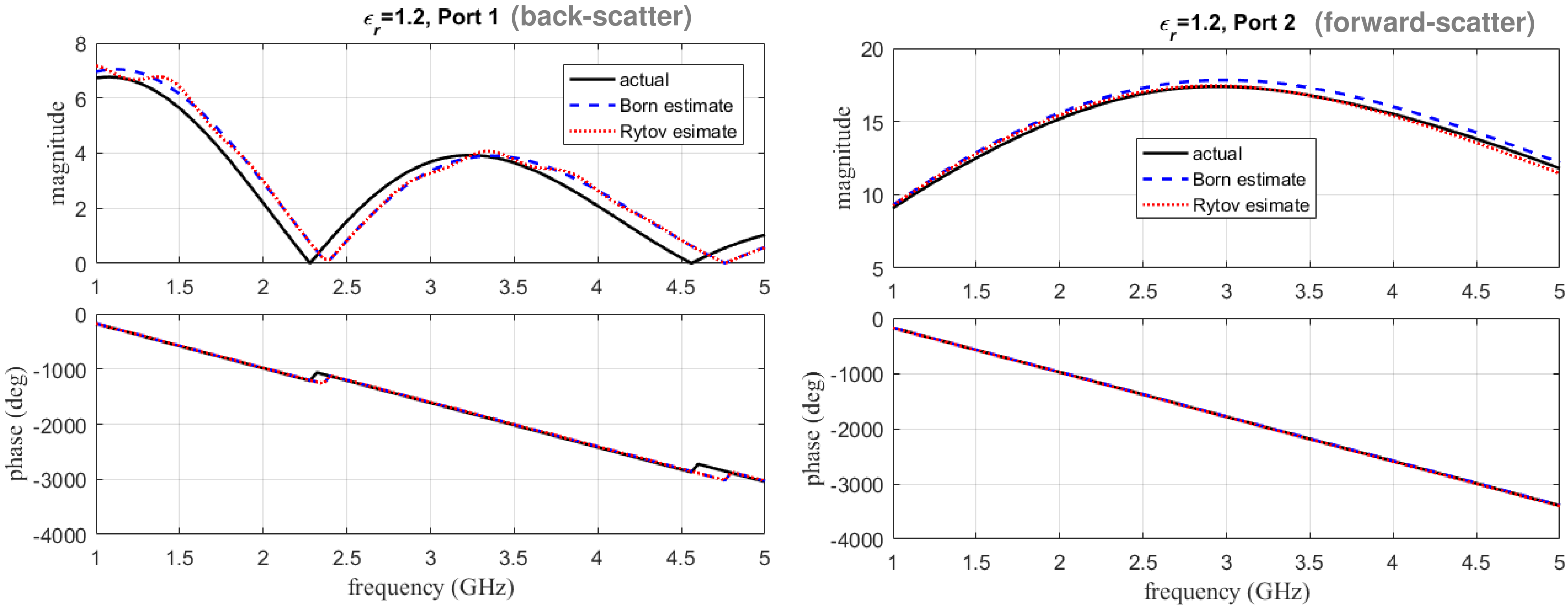
$$\left(\varepsilon_{\mathrm{r,s}}^{\mathrm{1GHz}}\right)_{\max} < 12.25$$

$$\left(\varepsilon_{\mathrm{r,s}}^{\mathrm{5GHz}}\right)_{\max} < 2.25$$

- notice independence of electrical size

EXAMPLE: BORN vs. RYTOV APPROXIMATION OF THE DATA – 2

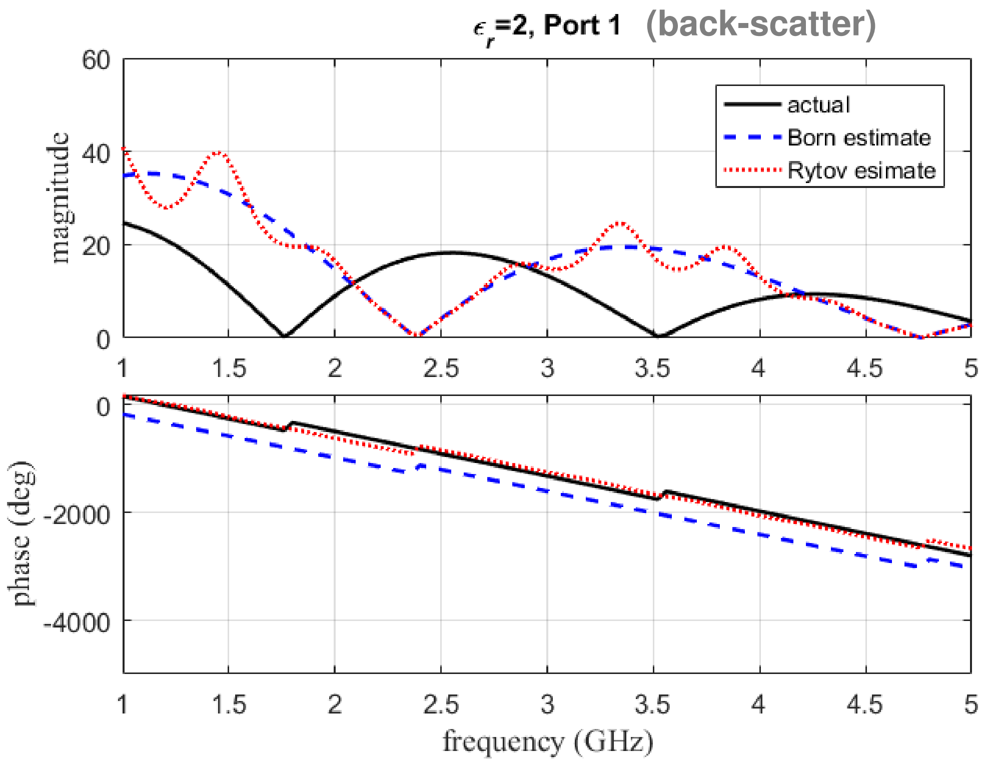
slab relative permittivity = 1.2

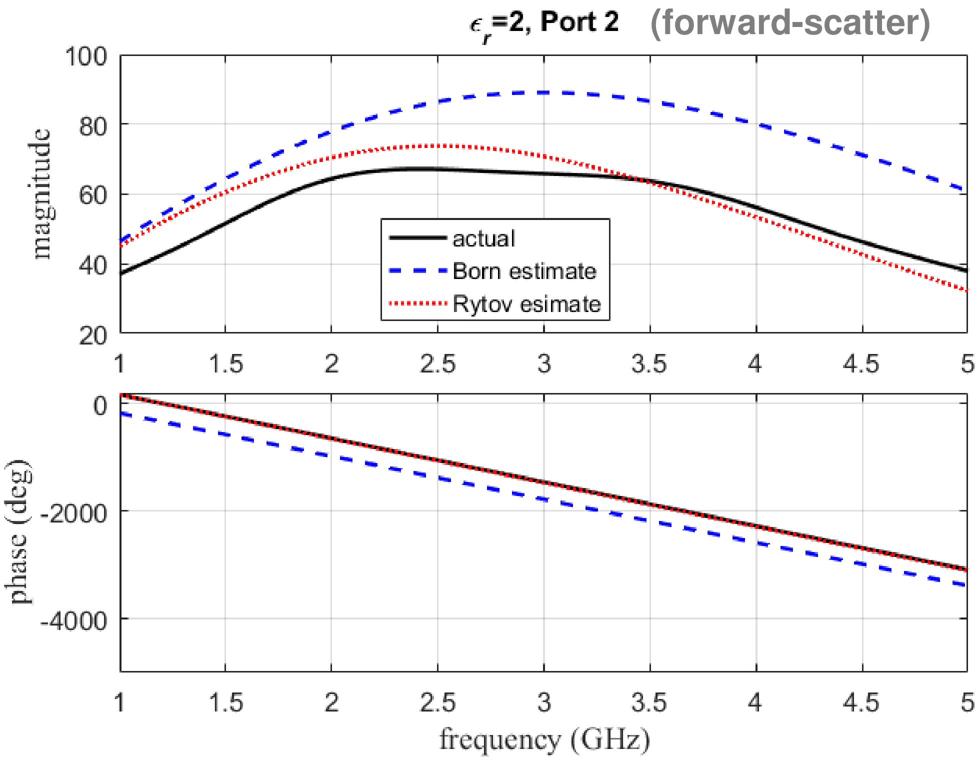


- both approximations perform very well: BA slightly better on back-scatter, RA slightly better on forward-scatter

EXAMPLE: BORN vs. RYTOV APPROXIMATION OF THE DATA – 3

slab relative permittivity = 2.0

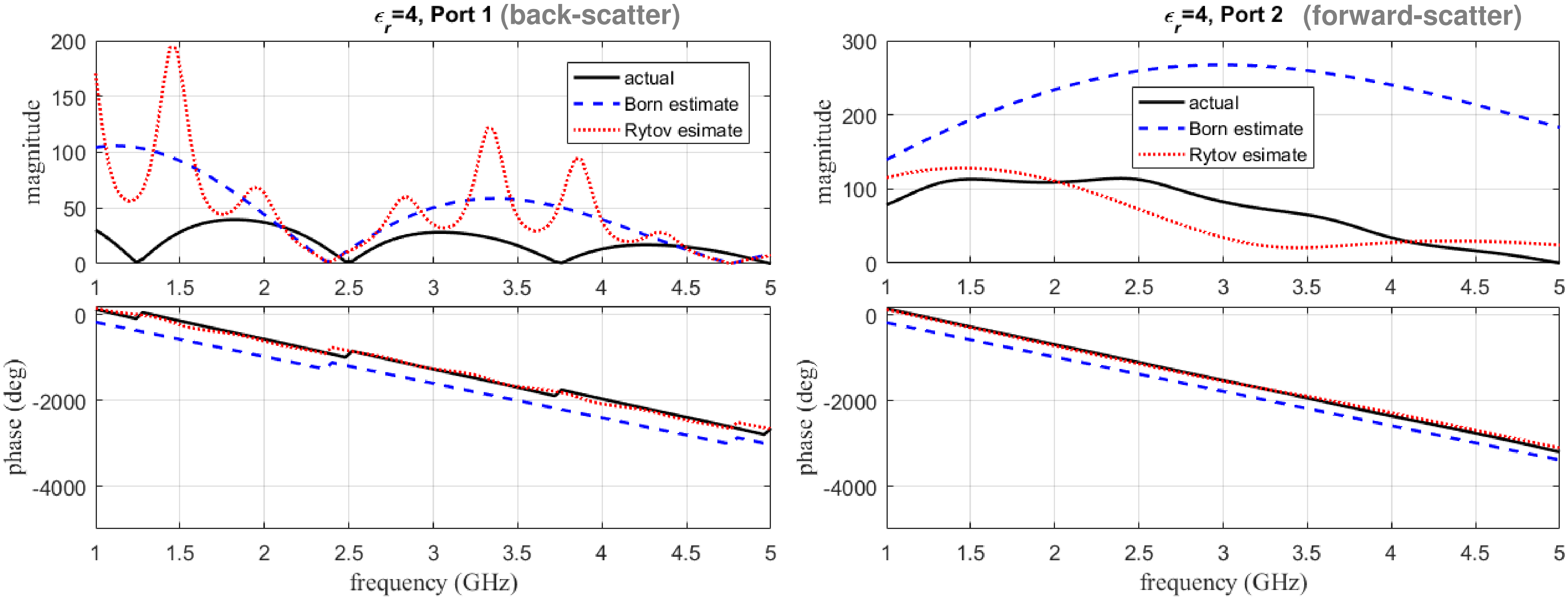




- both approximation show errors in magnitude of back-scatter
- RA better on forward-scatter and in the phase of back-scatter

EXAMPLE: BORN vs. RYTOV APPROXIMATION OF THE DATA – 4

slab relative permittivity = 4.0



- both approximation show large errors in magnitude of back-scatter
- RA much better on forward-scatter and the phase of both back- and forward scatter

ROLES OF DATA AND STATE EQUATIONS IN IMAGE RECONSTRUCTION

FORWARD MODEL:

$$\underbrace{F(\boldsymbol{x})=\boldsymbol{d}}_{\text{data equation}} \quad \underbrace{\mathcal{L}_{\mathrm{ME}}\{\boldsymbol{x},\boldsymbol{E}\}=\boldsymbol{E}}_{\text{state equations}}$$

?

INVERSION:

$$\underbrace{\boldsymbol{x}=F^{-1}\{\boldsymbol{d}\}}_{\substack{\text{solving}\\ \text{data equation}}} \text{ subject to } \mathcal{L}_{\mathrm{ME}}\{\boldsymbol{x},\boldsymbol{E}\}=\boldsymbol{E}$$

data equation:

$$\underbrace{\mathbf{E}^{\mathrm{sc}}(\mathbf{r}\in S_{\mathrm{a}})}_{\text{data } \boldsymbol{d} \text{ (external field)}} = \iiint_{V_{\mathrm{s}}} K(\mathbf{r}')\underline{\underline{\mathbf{G}}}_{\mathrm{b}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}^{\mathrm{tot}}(\mathbf{r}')d\mathbf{r}'$$

- the unknown is the contrast
- ensures that for a given internal field the forward model matches the data

state equation:

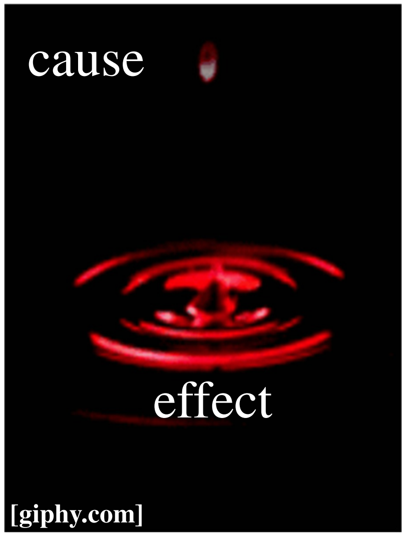
$$\underbrace{\mathbf{E}^{\mathrm{tot}}(\mathbf{r}\in V_{\mathrm{s}})}_{\text{internal field}} = \iiint_{V_{\mathrm{s}}} K(\mathbf{r}')\underline{\underline{\mathbf{G}}}_{\mathrm{b}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}^{\mathrm{tot}}(\mathbf{r}')d\mathbf{r}'$$

- the unknown is the internal field
- ensures that for a given contrast the internal field satisfies Maxwell’s eqns.

- image reconstruction is an interplay of the two forward-model equations
- the complete inverse problem solution involves determining not only the contrast but also the total internal field

FORWARD *vs.* INVERSE PROBLEM

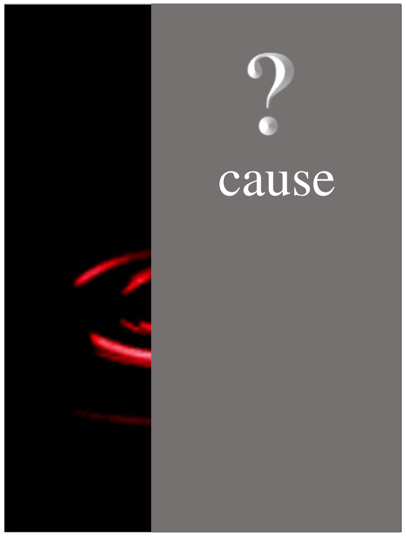
forward problem



[giphy.com]

- from cause to effect
- **unique solution**

inverse problem



- from effect to cause
- **not unique**

FORWARD *vs.* INVERSE PROBLEM IN EM SCATTERING

EM ANALYSIS

- **cause known**
  - excitation
  - boundary conditions
  - medium properties
- **effect unknown**
  - scattering parameters
  - radar cross-section
  - antenna far-field pattern
  - etc.

General-purpose forward solvers exist – EM simulators

INVERSE SCATTERING

- **cause – known aspects**
  - excitation (antennas)
  - boundary conditions
- **cause – unknown aspects**
  - medium properties
- **effect known only partially**
  - scattering parameters
  - radar return

General-purpose inverse solvers do not exist

RYTOV'S APPROXIMATION OF THE TOTAL FIELD

- Rytov's approximation of the ***total*** field is an exponential correction to the incident field

[S.M. Rytov, *Izv. Akad. Nauk SSSR*, Ser. Fiz 2 (1937)]

- the total field is represented as

$$U_{\mathrm{R}}^{\mathrm{tot}}(\mathbf{r})=U^{\mathrm{inc}}(\mathbf{r})\exp\left[U^{\mathrm{sc}}(\mathbf{r})/U^{\mathrm{inc}}(\mathbf{r})\right]$$

- 0th-order Rytov approximation ($U^{\mathrm{sc}}\approx 0$) – used to approximate total ***internal*** field

$$U_{\mathrm{R}}^{\mathrm{tot}}(\mathbf{r})\approx U^{\mathrm{inc}}(\mathbf{r}) \longleftarrow \textit{same as } 0^{th}\textit{-order Born approximation}$$

- 1st-order Rytov approximation – used to approximate total ***external*** field (data)

$$U_{\mathrm{R}}^{\mathrm{tot}}(\mathbf{r})=U^{\mathrm{inc}}(\mathbf{r})\cdot\exp\left\{\left[U^{\mathrm{inc}}(\mathbf{r})\right]^{-1}\underbrace{\iiint_{V_{\mathrm{s}}}G_{\mathrm{b}}(\mathbf{r},\mathbf{r}')\,K(\mathbf{r}')U^{\mathrm{inc}}(\mathbf{r}')dv'}_{U_{\mathrm{B}}^{\mathrm{sc}}(\mathbf{r})\text{ 1st order BA of scattered field}}\right\}$$

or

$$U_{\mathrm{R}}^{\mathrm{tot}}(\mathbf{r})=U^{\mathrm{inc}}(\mathbf{r})\exp\left[U_{\mathrm{B}}^{\mathrm{sc}}(\mathbf{r})/U^{\mathrm{inc}}(\mathbf{r})\right]$$

RYTOV'S APPROXIMATION OF THE SCATTERED-FIELD DATA

- Rytov's approximation of *S*-parameter data

$$S_{ik}^{\mathrm{tot}} \approx S_{ik}^{\mathrm{inc}} \exp\left[(S_{ik}^{\mathrm{sc}})_{\mathrm{R}} / S_{ik}^{\mathrm{inc}} \right] \Rightarrow \boxed{(S_{ik}^{\mathrm{sc}})_{\mathrm{R}} \approx S_{ik}^{\mathrm{inc}} \ln\left(\frac{S_{ik}^{\mathrm{tot}}}{S_{ik}^{\mathrm{inc}}} \right)}$$

  - ➢ compare with Born's data approximation $\boxed{(S_{ik}^{\mathrm{sc}})_{\mathrm{B}} \approx S_{ik}^{\mathrm{tot}} - S_{ik}^{\mathrm{inc}}}$

- limitation of the Rytov's approximation of the data

$$\boxed{\left(k_{\mathrm{s}}^2 - k_{\mathrm{b}}^2\right) / k_{\mathrm{b}}^2 < 1} \quad \text{or} \quad \boxed{\left(\varepsilon_{\mathrm{r,s}} - \varepsilon_{\mathrm{r,b}}\right) / \varepsilon_{\mathrm{r,b}} < 1}$$

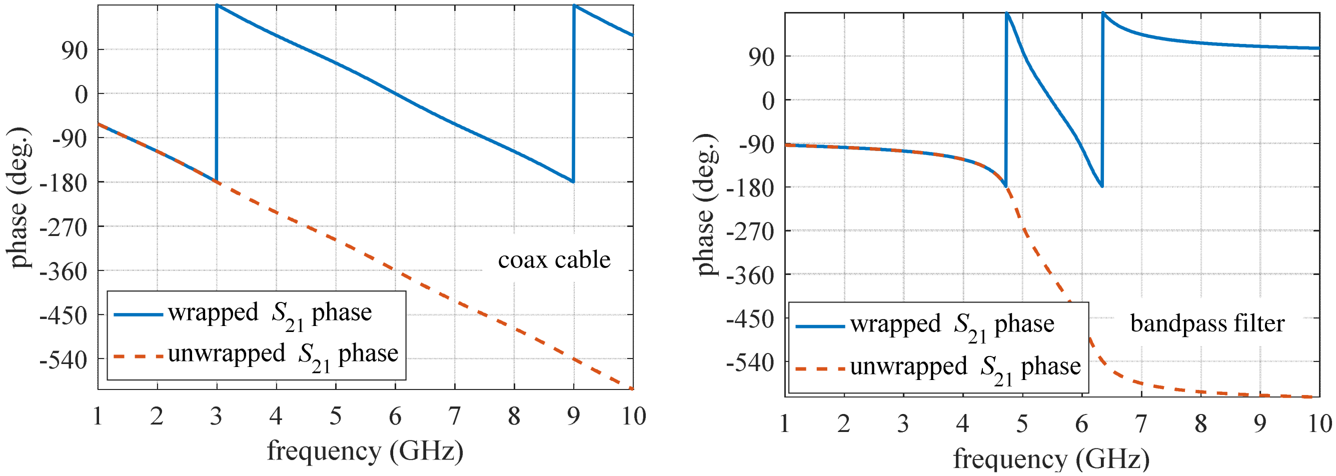
  - ➢ no limitation on the size of the scattering object – advantage over Born's approximation
  - ➢ very strict limitation on the relative contrast – disadvantage to Born's approximation

PITFALLS IN THE USE OF RYTOV'S APPROXIMATION

1) Rytov's approximation is prone to phase errors

must be unwrapped

$$(S_{ik}^{\mathrm{sc}})_{\mathrm{R}} = S_{ik}^{\mathrm{inc}} \ln\left(\frac{S_{ik}^{\mathrm{tot}}}{S_{ik}^{\mathrm{inc}}}\right) \Rightarrow \frac{(S_{ik}^{\mathrm{sc}})_{\mathrm{R}}}{S_{ik}^{\mathrm{inc}}} = \left\{\ln\frac{|S_{ik}^{\mathrm{tot}}|}{|S_{ik}^{\mathrm{inc}}|} + \mathrm{i}\boxed{\left(\angle S_{ik}^{\mathrm{tot}} - \angle S_{ik}^{\mathrm{inc}}\right)}\right\}$$

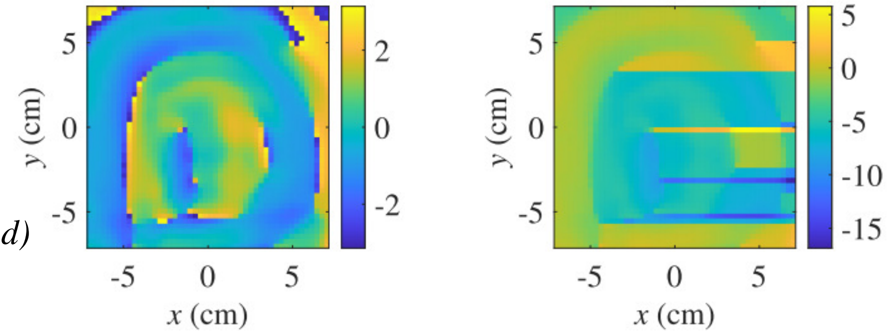


phase unwrapping in the frequency domain ensures smooth behavior in this domain

- ➢ unwrapped data in frequency does <u>not</u> ensure continuity in space (over the acquisition surface) → phase discontinuity across the synthetic aperture corrupts inversion!

PITFALLS IN THE USE OF RYTOV'S APPROXIMATION – 2

example of discontinuous spatial distribution of measured $\angle S_{21}^{\text{tot}}$ (rad)



original (wrapped)

failed unwrapping in x and y

[Tajik *et al.*, *IEEE Trans. MTT* 2022]

2) Rytov's approximation is prone to errors when incident field (RO data) is weak

$$(S_{ik}^{\text{sc}})_{\text{R}} = S_{ik}^{\text{RO}} \ln\left(\frac{S_{ik}^{\text{OUT}}}{S_{ik}^{\text{RO}}}\right) \Rightarrow \frac{(S_{ik}^{\text{sc}})_{\text{R}}}{S_{ik}^{\text{RO}}} = \left\{\ln\frac{|S_{ik}^{\text{OUT}}|}{\boxed{|S_{ik}^{\text{RO}}|}} + \mathrm{i}\left(\angle S_{ik}^{\text{OUT}} - \angle S_{ik}^{\text{RO}}\right)\right\}$$

division by zero or noisy signal!

- best use in transmission measurement with significant incident-field signal strength
- safe to use for object thickness D such that

$$|k_{\text{s}} - k_{\text{b}}|\, D \ll 2\pi \Rightarrow \left|\sqrt{\varepsilon_{\text{r,s}}} - \sqrt{\varepsilon_{\text{r,b}}}\right| D / \lambda_0 \ll 2\pi$$